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Trajectories of particles at the surface of steep solitary waves

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Theoretical calculations show that the horizontal displacement of particles in the surface of steep solitary waves exceeds that predicted by the Korteweg–de Vries equation by as much as 100 %.

Experimental evidence is given in support of the higher values.

1. Introduction

The Great Solitary Wave of Scott Russell (1838, 1845†) is the simplest type of finite-amplitude wave of permanent form propagating in shallow water. Its discovery has stimulated many interesting theoretical investigations since the time of Boussinesq (1871) and Rayleigh (1876). For a recent review see Miles (1980).

Solitary waves of low amplitude are described approximately by the equation of Korteweg & de Vries (1895), which applies also to waves in other types of media, and it is sometimes suggested (see for example Miles 1980, p. 21) that more accurate theory offers little or no practical advantage over the Rayleigh–Boussinesq or the Korteweg–de Vries approximation. As regards the phase-speed of solitary waves this is certainly true, at least over the lower range of wave steepness, as was shown by Longuet-Higgins & Fenton (1974). But in other respects, such as the total 'drift' or horizontal displacement of a particle during the passage of a wave, the situation is quite otherwise. Fenton (1972) already found values for the surface drift which were considerably greater than those given by the Rayleigh–Boussinesq theory. Fenton's calculations lose accuracy as the highest wave is approached. But a recent calculation by Longuet-Higgins (1979) for solitary waves of maximum amplitude suggests that the total surface drift can exceed the Rayleigh–Boussinesq value by a factor greater than 2 (see figure 1 below).

Since the surface drift is a quantity of some interest for coastal engineers and others in estimating the degree of horizontal mixing in the surf zone and just outside, it seemed to the author worth while to carry out some experiments to test the theoretical calculations. In spite of the inherent difficulties in generating steep solitary waves, which are discussed in § 3 below, these experiments, described in §§ 4 and 5, will be seen to support the higher values of the drift.

† See p. 332, line 32. Russell also used the terms 'Wave of translation' and 'Wave of the first order'.

2. Theoretical estimates

In the Rayleigh–Boussinesq theory (see for example Lamb 1932, § 252), the surface elevation η in a solitary wave of amplitude a in water of undisturbed depth h is given by

$$\eta = a\,\mathrm{sech}^2\{\alpha(x-ct)/h\}, \tag{2.1}$$

where

$$\alpha^2 = \tfrac{3}{4}a/h, \tag{2.2}$$

and

$$c^2 = gh(1+a/h). \tag{2.3}$$

The horizontal particle velocity u is found from Bernoulli's equation

$$c^2-q^2 = 2g\eta, \tag{2.4}$$

where $q^2 = (u-c)^2+v^2$. So, to lowest order,

$$u = g\eta/c, \tag{2.5}$$

while the vertical velocity v is approximately $\partial\eta/\partial t$. Hence the horizontal and vertical displacements X and Y are given by

$$\begin{aligned} X/h &= \int (u/h)\,dt = -\int (u/ch)\,dx \\ &= -(a/h)^2\int \mathrm{sech}^2(\alpha x/h)\,dx \\ &= -\frac{2}{\sqrt{3}}(a/h)^{\frac{1}{2}}\tanh(\alpha x/h) \end{aligned} \tag{2.6}$$

and

$$Y/h = (a/h)\,\mathrm{sech}^2(\alpha x/h). \tag{2.7}$$

Therefore a particle trajectory is given parametrically by

$$X/h = -\frac{2}{\sqrt{3}}\tau, \quad Y/h = (a/h)-\tau^2, \tag{2.8}$$

where $\tau = (a/h)^{\frac{1}{2}}\tanh(\alpha x/h)$. This represents a parabola with a vertical axis, passing through the points $(X, Y) = (0, a)$ and $[\pm 2(a/h)^{\frac{1}{2}}/\sqrt{3}, 0]$. The radius of curvature at the vertex of the parabola is $\frac{2}{3}h$, so that the centre of curvature is at about $\frac{1}{3}h$ above the bottom.

The total displacements are given by

$$[X]/h = \frac{4}{\sqrt{3}}(a/h)^{\frac{1}{2}}, \quad [Y]/h = a/h. \tag{2.9}$$

The curve of $[Y]/h$ versus $[X]/h$ therefore has the simple equation

$$[Y]/h = \tfrac{3}{16}\{[X]/h\}^2. \tag{2.10}$$

This corresponds to the left-hand (broken) curve in figure 1. It intersects the line $[Y]/h = 0{\cdot}83$ (corresponding to waves of limiting steepness) where $[X]/h \doteqdot 2{\cdot}1$.

The solid curve in figure 1 corresponds to the ninth-order calculation of surface

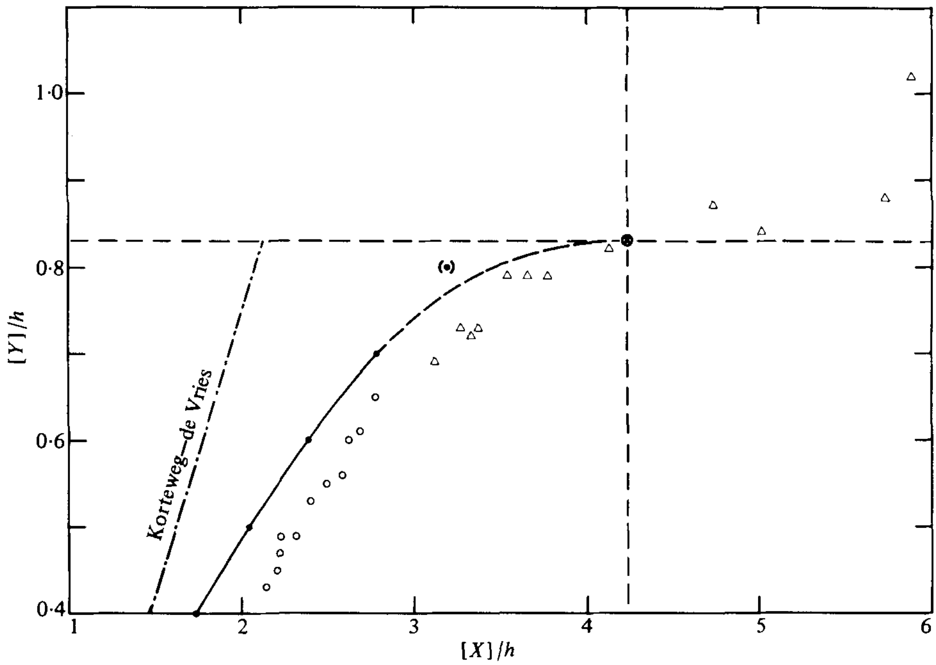


FIGURE 1. The total horizontal displacement $[X]$ versus the maximum vertical displacement $[Y]$ at the surface of a solitary wave in water of undisturbed depth h. – · – · –, approximate theory of Boussinesq and Rayleigh; —●—, ninth-order theory of Fenton (1972); ⊗, calculation by Longuet-Higgins (1979) for wave of greatest height; △, ○, observed total displacements at distances a = 3·5 m (△) and x = 6·1 m (○) from wavemaker.

drift' by Fenton (1972). The solid circular plots are taken from the upper curve in Fenton's figure 4. The calculation loses accuracy for $a/h > 0{\cdot}7$, as was shown by Longuet-Higgins & Fenton (1974). Accordingly the solid part of the curve in figure 1 extends only up to this point.

The isolated theoretical point at $[Y]/h = 0{\cdot}83$, $[X]/h = 4{\cdot}27$ corresponds to the calculation by Longuet-Higgins (1979) for the solitary wave of maximum amplitude. The calculated particle trajectory is shown in figure 7 of the same paper.

From figure 1 it is clear that the accurate value of the surface drift exceeds that given by the Rayleigh–Boussinesq theory by between 25 and 40 per cent when a/h lies between 0·4 and 0·7, and by as much as 100 per cent at higher values of a/h.

In contrast to the surface drift, the bottom drift for the steepest wave, as calculated by Price (1971), is only $1{\cdot}7h$. This is less than is given either by Rayleigh–Boussinesq theory ($2{\cdot}1h$) or by a second approximation due to Laitone (1960) which yields $2{\cdot}8h$.

3. The generation of solitary waves

To generate a solitary wave in the laboratory presents some interesting difficulties. Scott Russell (1845) used the simple method of allowing a solid weight to fall from near the surface to the bottom of the tank, displacing a known volume of fluid. The success of this method, at least for waves of moderate steepness, is due to the fact that the speed of a solitary wave is, according to (2·3), an increasing function of its amplitude a. Hence the wave tends to outrun any transient disturbance associated with its generation, and also any other solitary waves of lower amplitude that may be generated at the same time (see Hammack & Segur 1974).

Two similar methods were used by Daily & Stephan (1952). In the first, they displaced a given mass of water by the vertical motion of a piston rising from the bottom of the tank. In the second method they released a known mass of water behind a vertical, moveable barrier at one end of the tank. These methods worked best at intermediate wave amplitudes. For very low waves they fail on account of the extreme length of solitary waves of low amplitude. On the other hand at wave steepnesses a/h exceeding about 0·6 they failed for a different reason, which was not then apparent.

It is now known that, as the amplitude of a solitary wave increases, its phase speed increases only up to a certain value and then falls slightly, as was first shown by Longuet-Higgins & Fenton (1974). Moreover the total displaced mass, considered as a function of the wave amplitude, behaves in a similar way. Hence, over the upper range of wave amplitudes there actually exist two different waves with the same total displaced mass. For a given displacement the fluid must 'choose' between two distinct motions. Which shall it choose?

It seems that generally the fluid selects the wave of lower amplitude. The reason is evidently related to the stability of the corresponding motion. For, not only the mass but also the total energy has a maximum, at about the same wave steepness ($a/h \doteqdot 0{\cdot}75$). If a wave is steeper than this critical value, any dissipation within the fluid or at the boundaries will cause it to steepen further and ultimately to break; whereas a wave of less than the critical steepness can simply lose height gradually.

A further but immediate consequence is that a very steep solitary wave is most unlikely to be attained except in a more-or-less transient state.

Conceivably one could construct a wave generator to follow very closely the orbital motion in a steep solitary wave. But a plane vertical plate would not be sufficient. The wavemaker would have to be designed so that the displacement at the surface exceeded that at the bottom by a factor as great as 2·5.

So far we have considered only water of uniform depth. By making a wave run into gently shoaling water one might hope to steepen the wave gradually to the point where it overtopped the energy curve (corresponding to the local depth) and then ran down the curve towards breaking (see Longuet-Higgins & Fenton 1974, § 9). Thus it might be possible to 'catch' a steep solitary wave in an almost steady state.

One disadvantage, however, in using a non-uniform depth of water is that any initially pure, symmetric solitary wave then becomes asymmetrical and develops a long 'tail' following the main elevation (see Kaup & Newell 1978) or may produce an additional soliton (Madsen & Mei 1969).

Conceivably one might overcome this difficulty by a judicious combination of wavemaker, surface slope and frictional dissipation. However in the experiments described below our aim was more modest: to observe the particle trajectories at the free surface in a motion that was approximately that of a solitary wave, and to compare the results with the theory of § 2.

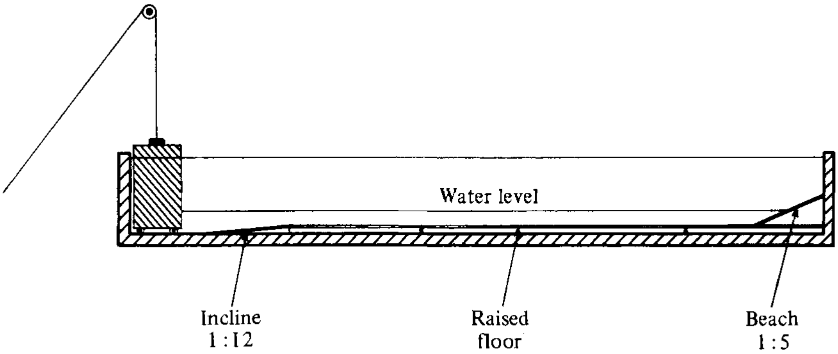


FIGURE 2. Sketch of wave tank, wavemaker and beach (not to scale).

4. Experimental apparatus

The experiments were carried out in a steel and Plexiglas tank 9·7 m long and 30 cm wide, as sketched in figure 2. The maximum depth of water was 13 cm. At a distance 1·3 m to 1·9 m from the wave generator a plane ramp, with slope 1:12, led to a horizontal floor raised 5·6 cm above the bottom of the tank. At the far end was a plane beach, of length 60 cm and slope 1 : 5, covered with wire gauze, to act as an absorber.

The wavemaker was essentially similar to that of Scott Russell (1845). A hollow rectangular block, 25 cm in length and 43 cm high, was guided vertically by two steel rods attached to the near end of the tank, as in figures 2 and 3. Two small buffers attached to the bottom of the block ensured a minimum gap of 1 cm between the block and the bottom of the tank. In operation, the block was first raised as shown in figure 3 (a) so that the lower surface was at a predetermined height above the bottom. The disturbance having died down, the block was then released magnetically, so as to fall to the bottom under its own weight (figure 3b).

The resulting disturbance was propagated down the tank. On travelling up the ramp the leading wave became steepened (figure 4) before passing on down the tank to be dissipated on the beach at the far end.

The waves were observed in the central section of the tank, at distances 3·5 m and 6·1 m from the wavemaker. To record the trajectories, a small wooden bead, painted white and loaded so as to be slightly buoyant, was placed on the water surface and illuminated from above. During passage of the wave a time-exposure of the track, seen against a dark background, was recorded by a camera distant 1·6 m from the near side of the tank.

At the same time a record was made of the surface elevation at neighbouring fixed point, by a capacitance-wire wave recorder.

5. Results of the experiments

Figure 5 shows a typical time-trace of the surface elevation at a distance $x = 3{\cdot}5$ m from the front face of the wavemaker. This reveals a primary wave, of near maximum steepness, followed by a much lower secondary wave. At greater distances from the wavemaker the primary and secondary waves become separated further, but at the cost of dissipating some of the energy in the primary wave by friction at the bottom and sides of the tank.

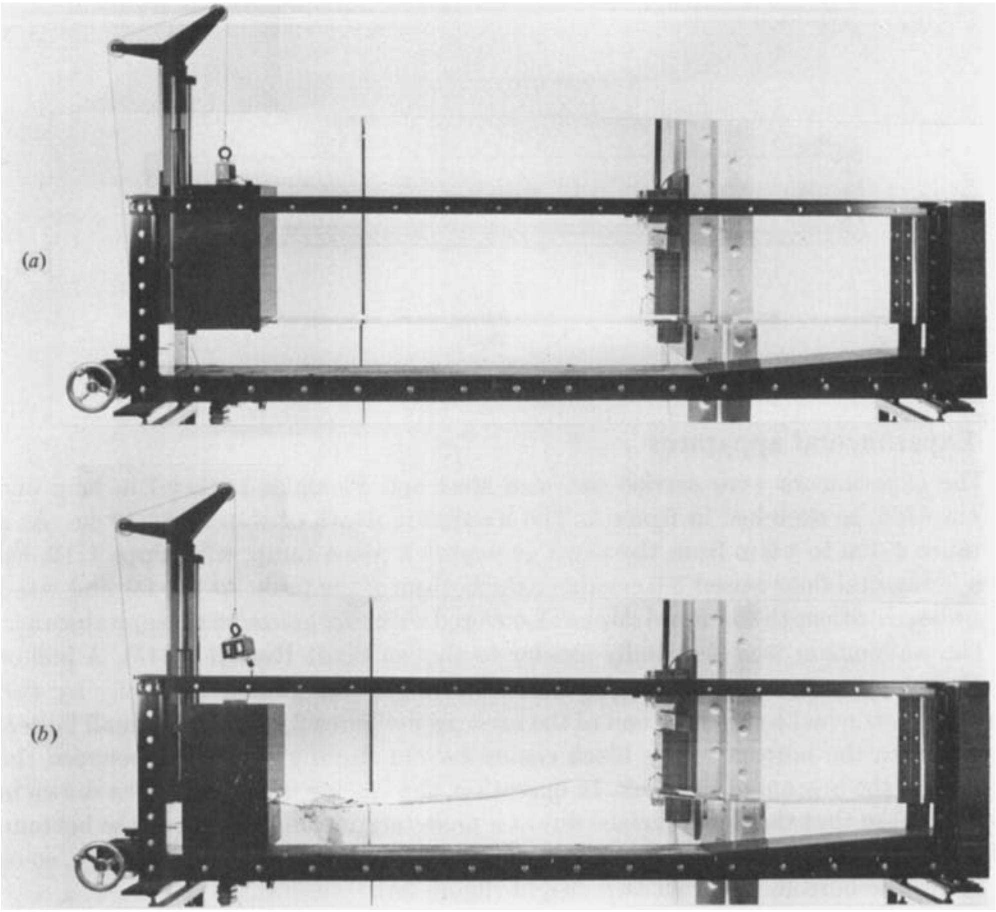


FIGURE 3. The near end of the channel, showing the wavemaker block (a) before release, (b) after release.

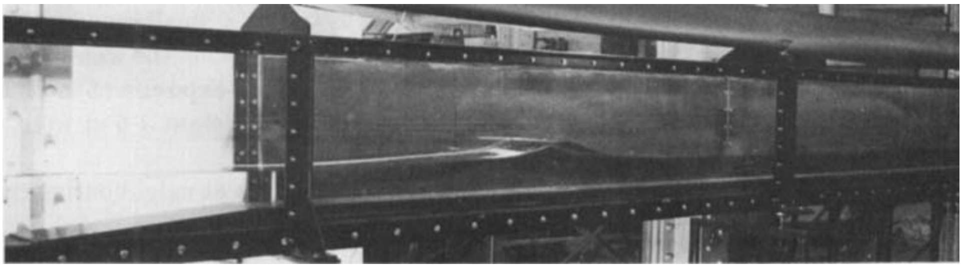


FIGURE 4. The central section of the channel, showing the same wave as in figure 3(b) at a later stage.

Figure 6(a) shows the trajectory of the bead in the wave corresponding to figure 5. The initial position of the bead is on the left, and the near edge of the 'floor' is visible below. The bead rises in a broad arc and then falls to almost the same level on the right. This is followed by a smaller, secondary arc, evidently associated with the 'tail' or secondary elevation in figure 5.

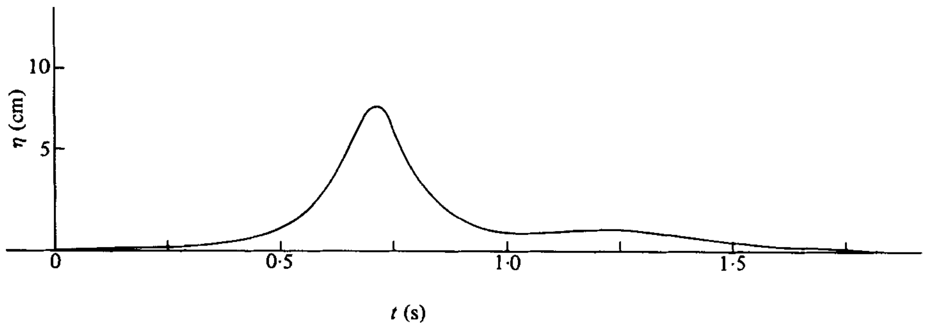


FIGURE 5. Record of the surface elevation at $x = 3{\cdot}5$ m, when $a/h = 0{\cdot}69$.

Figure 6(*b*) shows the trajectory in a wave that is just breaking, and figure 6(*c*) corresponds to a wave that has a definite 'roller' or whitecap at the crest. In the latter case the particle is swept along in a much broader and irregular path. Over part of the trajectory the light is obscured by turbulence or ripples at the free surface.

In analysing the trajectories it was assumed that the primary wave could be regarded as a solitary wave propagated in a depth of water modified by the presence of the secondary wave. Accordingly the total displacement $[X]$ was taken as the horizontal distance between the initial position P of the bead and the next position Q of minimum elevation. The mean depth h was taken as the height of the mid-point M of PQ above the bottom, and the vertical displacement $[Y]$ was taken as the maximum height of the trajectory above M.

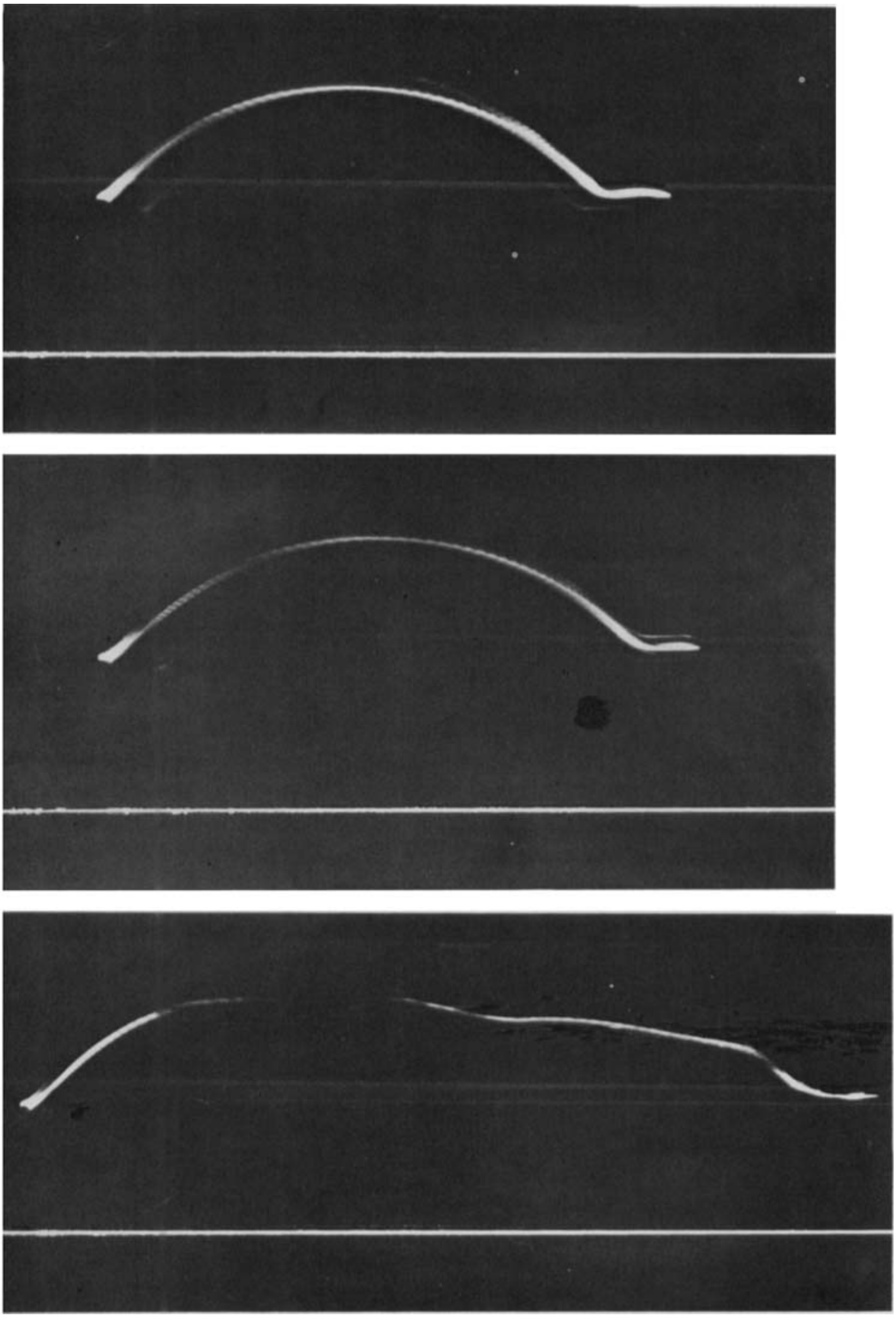
The measured values of $[X]/h$ are shown plotted against $[Y]/h$ in figure 1.

6. Discussion and conclusions

In figure 1 the points in the lower left-hand quadrant correspond to non-breaking waves. Although it is not easy to judge the extent by which the distance $[X]$ is lengthened owing to the presence of the secondary wave, one would expect the increment to be certainly less than the length of the secondary 'hump' in the trajectory, which is typically 10 % of the uncorrected value of $[X]$. This is approximately the amount by which the measured value of $[X]$ exceeds the value given by Fenton's calculation (the solid curve in figure 1). It appears then that the plotted observations are closer to the full curve of Fenton (1972) than to the Rayleigh–Boussinesq curve. Moreover the trend of the observations passes close to the theoretical point calculated by Longuet-Higgins (1979) for waves of maximum steepness.

Points in the upper right-hand quadrant of figure 1 correspond to breaking waves. The scatter of such observations is relatively large, as might be expected from the turbulent nature of whitecaps (see Longuet-Higgins & Turner 1974).

The accuracy of the experiments reported here could probably be improved in the lower part of the range, say $a/h < 0{\cdot}65$. At higher values of a/h, however, it is doubtful whether solitary waves are in fact stable, for the reasons discussed in § 3. Hence any measurements must in practice be made on flows which are in an unsteady state. Our observations represent a first attempt to do this.



(a)

(b)

(c)

FIGURE 6. Typical particle trajectories at $x = 3{\cdot}5$ m, when (*a*) $a/h = 0{\cdot}69$ (non-breaking wave), (*b*) $a/h = 0{\cdot}84$ (limiting wave), (*c*) $a/h = 0{\cdot}92$ (breaking wave).

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